Fundamental theorems

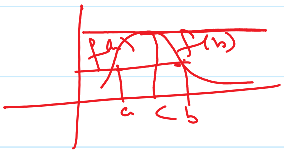
L.H.L = R.H.L = f(x.)

Rolle's Theorem :

If f(x) is

1) a continuous function on the interval [a, b]
2) differentiable on the open interval (a, b)
3) and f(a) = f(b),

then there is at least one value c of x in the interval (a, b) such that f '(c) = 0



c Roll's Point c $f'(c) = 0$ Roll's Point

① Poly cts $x, x^2, x^2+2x+1, x^{100}$
② Sinx, Cosx cts
③ exp. cts e^x, e^{x^2}, e^{x^2+2}
④ log(x), $x > 0$

log(−1) =

f, g cts

f(g(x)) cts ① f ± g cts ② f · g cts ③ $\frac{f}{g}$ cts but $g \neq 0$

$\tan x = \frac{\sin x}{\cos x}$ cts, Cos x cts, Cos x = 0 on

Cos x = 0, $x = \frac{\pi}{2}$

Sin θ = 0 ⇒ θ = nπ
Cos θ = 0 ⇒ θ = (2n+1)$\frac{\pi}{2}$
n = 0, ±1, ±2, ±3 ...

Sin(nπ) = 0
Cos(nπ) = $(-1)^n$

$\frac{d}{dx}|x| = \frac{x}{|x|}$

c Rolle's Pot $-2x+6$

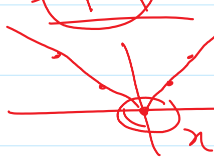
Example 1 The graph of $f(x) = -x^2 + 6x - 6$ for $1 \le x \le 5$ is shown below.

$a \le x \le b$

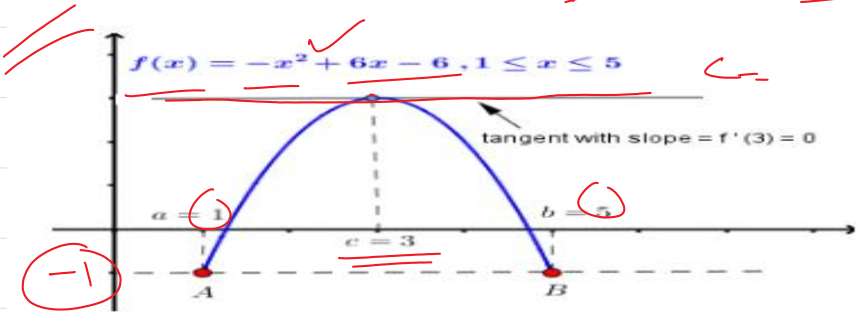
① cts
② diff
③ f(a) = f(b)
f(1) = f(5)
−1 + 6 − 6 = −25 + 30 − 6
−1 = −1

$f'(x) = -2x + 6$
$f'(c) = 0$
$-2c + 6 = 0$
$\boxed{c = 3}$

|x| cts but not diff



x = 0



f(1) = f(5) = - 1

and f is continuous on [1 , 5] and differentiable on (1 , 5)
hence,
according to Rolle's theorem, there exists at least one value of x = c such that f '(c) = 0.

f '(x) = - 2 x + 6

f '(c) = - 2 c + 6 = 0   c = 3

Solve the above equation to obtain

c = 3

Therefore at x = 3 there is a tangent to the graph of f that has a slope equal to zero (horizontal line)

Ex. The graph of f(x) = sin(x) + 2 for 0 ≤ x ≤ 2π is shown below.

$f'(c) = 0$

$\cos c = 0$

$c = (2n+1)\frac{\pi}{2}$

$n = 0, \pm 1, \pm 2, \ldots$

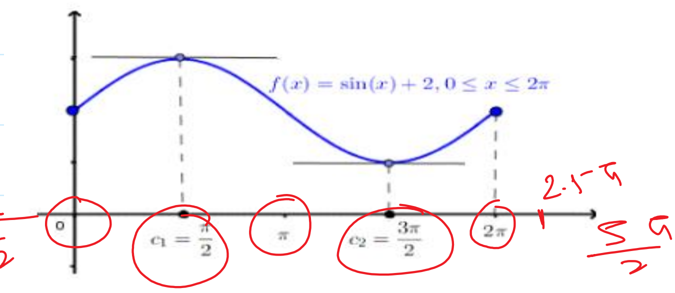
$0 \le \pi \le 2\pi$

if $n = 0$, $c = \pi/2$

if $n = 1$, $c = 3\pi/2$

if $n = -1$, $c = -\pi/2$

if $n = 2$, $c = 5\pi/2$



f(0) = f(2π) = 2 and f is continuous on [0 , 2π] and differentiable on (0 , 2π) hence, according to Rolle's theorem, there exists at least one value (there may be more than one!) of x = c such that f '(c) = 0.

f '(x) = cos(x)

f '(c) = cos(c) = 0

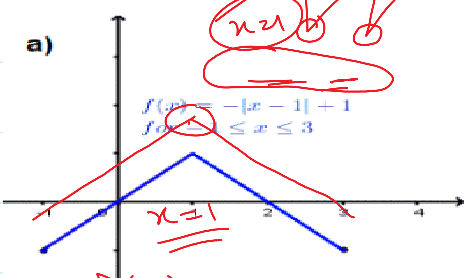
The above equation has two solutions on the interval [0 , 2π]

$c_1 = \pi/2$ and $c_2 = 3\pi/2$.

Therefore both at x = π/2 and x = 3 π/2 there are tangents to the graph that have a slope equal to zero

$f(x) = -|x-1|+1$

Ex. Function $f(x) = -|x-1| + 1, -1 \le x \le 3$, does not satisfy Rolle's theorem: although it is continuous and f(-1) = f(3), the function is not differentiable at x = 1 and therefore f '(c) = 0 with c in the interval (-1 , 3) is not guaranteed. In fact it is easy to see that there is no horizontal tangent to the graph of f on the interval (-1 , 3).



LHL = RHL = f(x₀)

cos nπ = $(-1)^n = (-1)^2 = 1$

Which of the functions given below satisfy all three conditions of Rolle's theorem?

1) cts
2) diff
3) f(a) = f(b)

a) f(x) = cos(x) , for x in [0 , 2π]

f(0) = f(2π)

b) g(x) = |x - 2| , for x in [0 , 4]

c) h(x) = 1 / x², for x in [-1 , 1]

d) k(x) = |sin(x)| , for x in [0 , 2π]

Ex. Check that function f(x) = x² - 4 x + 3 on the interval [1 , 3] satisfies all conditions of Rolle's theorem and then find all values of x = c such that f '(c) = 0.

c = 2

2x − 4 = 0

2c − 4 =

Ex. Check that function g(x) = cos(x) on the interval [- π/2 , 3π/2] satisfies all conditions of Rolle's theorem and

Ex. Check that function g(x) = cos(x) on the interval
[- π/2 , 3π/2] satisfies all conditions of Rolle's theorem and
then find all values x = c such that g '(c) = 0.

$(0, \pi)$

$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

$f'(c) = 0$

$-\sin c = 0$

$\sin c = 0$

$\boxed{c = n\pi}$

$n = 0, \pm 1, \pm 2, \ldots$

if $n = 0$ $\quad c = 0$

if $n = 1$ $\quad c = \pi$

if $n = -1$ $\quad c = -\pi$

For problems 1 – 4 determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for the given function and interval.

1. $f(x) = x^3 - 4x^2 + 3$ on $[0, 4]$
2. $Q(z) = 15 + 2z - z^2$ on $[-2, 4]$
3. $h(t) = 1 - e^{t^2 - 9}$ on $[-3, 3]$
4. $g(w) = 1 + \cos[\pi w]$ on $[5, 9]$

1. $f(x) = x^2 - 2x - 8$ on $[-1, 3]$
2. $g(t) = 2t - t^2 - t^3$ on $[-2, 1]$

1. Verify Rolle's theorem for (i) $f(x) = (x + 2)^3 (x - 3)^4$ in $(-2, 3)$.

(ii) $y = e^x (\sin x - \cos x)$ in $(\pi/4, 5\pi/4)$. (iii) $f(x) = x(x + 3)\, e^{-1/2x}$ in $(-3, 0)$.

(iv) $f(x) = \log \left\{ \dfrac{x^2 + ab}{x(a + b)} \right\}$ in (a, b).

$\log(x)$, $x > 0$ $\quad (a, b)$

4.) $f(x) = \log(x^2 + ab) - \log(x(a+b))$

$f'(x) = \dfrac{1}{x^2 + ab} \cdot 2x - \dfrac{1}{x(a+b)} \cdot (a+b)$

$f'(c) = \dfrac{2c}{c^2 + ab} - \dfrac{a+b}{c(a+b)} = 0$

$2c^2 a + 2c^2 b - ac^2 - bc^2 - c^2 b - ab^2 = 0$

$(c^2 + ab)(ca + cb)$

$c^2(a+b) - ab(a+b) = 0 \qquad c = \pm\sqrt{ab}$

(a) $a + b$
(b) $a \cdot b$
(c) $\pm\sqrt{ab}$
(d) $\frac{a}{b}$

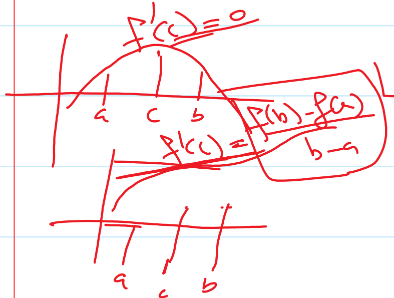
(iii)

$f(x) = x(x+3) \cdot e^{-\frac{1}{2}x} = (x^2 + 3x)\, e^{-\frac{1}{2}x}$

$f'(x) = (x^2 + 3x)\left(-\frac{1}{2}\right) e^{-\frac{1}{2}x} + (2x + 3)\, e^{-\frac{1}{2}x}$

$f'(c) = (c^2 + 3c)\left(-\frac{1}{2}\right) e^{-\frac{1}{2}c} + (2c + 3)\, e^{-\frac{1}{2}c} = 0$

① –
② –
③ $f(a) = f(b)$



$e^{-\frac{1}{2}c}\left((c^2 + 3c)\left(-\frac{1}{2}\right) + 2c + 3\right) = 0$

$-(c^2 + 3c) + 4c + 6 = 0$

$-c^2 + c + 6 = 0$

$c^2 - c - 6 = 0 \qquad c = -2, 3$

$\boxed{c = -2}$

① cts ② diff ③ $f(a) = f(b)$

$f(a) \neq f(b)$

$f(a) \neq f(b)$

Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following. $\quad f'(c)$

Mean Value Theorem

$f(a) \neq f(b)$

$f'(c) = 0$

$\frac{y_2 - y_1}{x_2 - x_1}$

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).

Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(c)(b-a) = f(b) - f(a)$

f'

$f(a)\ f(b)$

$f(b) - f(a) = f'(c)(b-a)$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

$f(a+h) - f(a) = h \cdot f'(a + \theta h)$

$a + h - a$

$[a, a+h]$

$a \le c \le b$

$a \quad a+h$

$a + \theta h$

$0 < \theta < 1$

$\frac{1}{4}$, $\theta = \frac{1}{2}$, $\frac{1}{3}$

Second form. If we write $b = a + h$, then since $a < c < b$,

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Thus the mean value theorem may be stated as follows :

If (i) f(x) is continuous in the closed interval $[a, a + h]$ and (ii) $f'(x)$ exists in the open interval $(a, a + h)$, then there is at least one number $\theta(0 < \theta < 1)$ such that

$$\mathbf{f(a + h) = f(a) + hf'(a + \theta h)}$$

$a \quad c \quad a+h$, $a + \frac{h}{2}$

$a + \theta \cdot h$

$a + \frac{1}{2}h$

Ex. Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1, 2]$$

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$f'(x) = 3x^2 + 4x - 1$

$f(2) = 8 + 8 - 2 = 14$

$f(-1) = -1 + 2 + 1 = 2$

$3c^2 + 4c - 1 = \frac{14 - 2}{3} = 4$

$c = \frac{-4 \pm \sqrt{16+60}}{}$

$3c^2 + 4c - 5 = 0$

$[-1, 2]$

$c = 0.78$

$c = \frac{-4 \pm 8.7}{6} = \frac{-12.7}{6}, \frac{4.7}{6}$

Ex. Suppose that we know that f(x) is continuous and differentiable on [6,15]. Let's also suppose that we know that f(6)=-2 and that we know that f'(x)≤10. What is the largest possible value for f(15)?

≤ 10

$f'(c) \le 8$

$f'(c) = 8$

$f(15) = 88$

$f(b) - f(a) = f'(c)(b-a)$

$f(b) = f(a) + f'(c)(b-a)$

$f(15) = f(6) + f'(c)(9)$

$f(15) = -2 + f'(c) \cdot 9$

$\le -2 + 90$

$f(15) \le -2 + 72$

$f(15) \le 70$

$f'(x) = 2$

$f'(x)$ $\frac{9}{10}$

$f(15) \le 88$

Ex. Determine all the number(s) c which satisfy the conclusion of the Mean Value Theorem for the given function and interval.

$$h(z) = 4z^3 - 8z^2 + 7z - 2 \text{ on } [2,5]$$

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$h(2) = 12 \qquad h(5) = 333 \qquad h'(z) = 12z^2 - 16z + 7$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$12c^2 - 16c + 7 = \frac{333 - 12}{5 - 2} = 107 \quad \rightarrow \quad 12c^2 - 16c - 100 = 0$$

$$c = \frac{2 \pm \sqrt{79}}{3} = -2.2961, \quad 3.6294$$

So, we found two values and, in this case, only the second is in the interval and so the value we want is,

$$\boxed{c = \frac{2 + \sqrt{79}}{3} = 3.6294}$$

$$A(t) = 8t + e^{-3t} \text{ on } [-2,3]$$

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$A(-2) = -16 + e^6 \qquad A(3) = 24 + e^{-9} \qquad A'(t) = 8 - 3e^{-3t}$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$8 - 3e^{-3c} = \frac{24 + e^{-9} - (-16 + e^6)}{3 - (-2)} = -72.6857$$

$$3e^{-3c} = 80.6857$$

$$e^{-3c} = 26.8952$$

$$-3c = \ln(26.8952) = 3.29195 \quad \Rightarrow \quad c = -1.0973$$

So, we found a single value and it is in the interval and so the value we want is,

$$\boxed{c = -1.0973}$$

$\cancel{f(b) = f(a) + f'(c)(b-a)}$

$f(x)$ cts diff $\quad [-7,0]$, $f(-7) = -3$, $f'(x) \le 2$

Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

$f(0) = f(-7) + f'(c)(0+7)$

$\le -3 + 2(7) \quad -3 + 14 = 11$

$f(0) \le -3 + 2(7) = 11$

Show that $f(x) = \cancel{x^3 - 7x^2 + 25x} + 8$ has ~~exactly one real root.~~ $\quad x^3$

(3) $f(0) = 0 - 0 + 0 + 8 = +ve$

$f(1) = 1 - 7 + 25 + 8 = +ve \qquad$ one

$f(-1) = -1 - 7 - 25 + 8 = -ve \qquad$ one $\quad$ +ve −ve!

$f(-1) = -1-7-25+8 = -ve$

$[-1, 0]$ Real

one

$-ve$

a, b $f(a) = 0$ $f(b) = 0$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = 0$

$3c^2 - 14c + 25 = 0$

$c = 14 \pm \sqrt{196-300}$ (-ve)

$\alpha \pm i\beta$

$\frac{1}{1+c^2}$

For problems 5 – 8 determine all the number(s) c which satisfy the conclusion of the Mean Value Theorem for the given function and interval.

5. $f(x) = x^3 - x^2 + x + 8$ on $[-3, 4]$

6. $g(t) = 2t^3 + t^2 + 7t - 1$ on $[1, 6]$

7. $P(t) = e^{2t} - 6t - 3$ on $[-1, 0]$

8. $h(x) = 9x - 8\sin\left(\frac{x}{2}\right)$ on $[-3, -1]$

(i) $f(x) = (x-1)(x-2)(x-3)$ **in** $(0, 4)$

(ii) $f(x) = \sin x$ **in** $[0, \pi]$

(iii) $f(x) = \log_e x$ **in** $[1, e]$.

(iv) $f(x) = e^x$ **in** $[0, 1]$.

Example 4.15. *Prove that (if* $0 < a < b < 1$*),* $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$.

Hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$. *(Mumba*

$f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{1+x^2}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$= \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$

$\frac{b-a}{1+b^2}$ $\frac{b-a}{1+a^2}$

b^2 a^2

$a < c < b$

$a^2 < c^2 < b^2$

$0 < a < b < 1$

$2 < 3$

$\frac{1}{2} > \frac{1}{3}$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

$(b-a)$

$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$

$\frac{b-a}{\ldots} > \tan^{-1} b - \tan^{-1} a > b-a$

If $f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ when $h = 1$ and $f(x) = (1 - x)^{5/2}$.

Example 4.16. *Prove that $\log(1 + x) = x/(1 + \theta x)$, where $0 < \theta < 1$ and hence deduce that*

$$\frac{x}{1+x} < \log(1+x) < x,\ x > a$$ (M...

Solution. Let $f(x) = \log(1 + x)$, then by second form of Lagrange's mean value theorem

$$f(a + h) = f(a) + h f'(a + \theta h), \qquad (0 < \theta < 1)$$

we have $\quad f(x) = f(0) + x f'(\theta x) \qquad$ [Taking $a = 0$, $h = x$]

or $\quad \log(1 + x) = \log(1) + x \,.\, 1/(1 + \theta x) \qquad [\because\ f'(x) = 1/(1 + x)]$

Hence $\quad \log(1 + x) = x/(1 + \theta x) \qquad$ ...(i) $[\because\ \log(1) = 0$

Since $\quad 0 < \theta < 1, \quad \therefore \quad 0 < \theta x < x$ for $x > 0$.

or $\quad 1 < 1 + \theta x < 1 + x \quad$ or $\quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$

or $\quad x > \frac{x}{1+\theta x} > \frac{x}{1+x}$

or $\quad \frac{x}{1+x} < \log(1+x) < x,\ x > 0. \qquad$ [By (i)]

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + h f'(a+\theta h)$

$e^x = 1 + x + \frac{x^2}{2!} \ldots x^n$

$f(a+h) \ldots$ $e^x = 1 + x \ldots$

Taylor's Theorem

If (i) f(x) and its first (n – 1) derivatives be continuous in [a, a + h], and (ii) $f^n(x)$ exists for every value of x in (a, a + h), then there is at least one number θ (0 < θ < 1), such that

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^n(a + \theta h)} \quad \ldots(1)$$

which is called Taylor's theorem with Lagrange's form remainder, the remainder R_n being $\frac{h^n}{n!} f^n(a + \theta h)$.

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Cor. 1. Taking $n = 1$ in (1), *Taylor's theorem reduces to Lagrange's Mean-value theorem.*

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(\theta x).$$

which is known as *Maclaurin's theorem with Lagrange's form of remainder.*

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f(x) = f(0) + x \cdot f'(0) + \ldots + x^n$

$e^x = e^0 + x \cdot e^0 + \frac{x^2}{2!} e^0 + \ldots + x^n (\theta x)$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} e^{\theta x}$

$f^n(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) \ldots + \frac{x^n}{n!} f^n(\theta x)$

$f(x) = \cos x$ $\quad f(0) = 1$

(-1)

Example 4.18. *Find the Maclaurin's theorem with Lagrange's form of remainder for f(x) = cos x.* (J.N.T.U.,

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$\cos x = 1$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = 1$

$f^{v}(x) = -\sin x$ $\quad f^{v}(0) = 0$

$f^{vi}(x) = -\cos x$ $\quad f^{vi}(0) = -1$

$-ve \quad +ve \quad -ve$

$1 + \frac{x^6}{6!}(-1) + \ldots + \frac{x^{2n}}{(2n)!} \cos(\theta x)(-1)$

$2n \quad 0, 1, 2, 3 \ldots$

$(-1)^0 = 1 \quad (-1)^n = 1$

$(-1)^2 = +1 \quad (-1)^1 = -$

$(-1)^2 \quad f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$

Example 4.19. *If $f(x) = \log(1+x)$, $x > 0$, using Maclaurin's theorem, show that for $0 < \theta < 1$,*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}.$$

Deduce that $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ *for* $x > 0$. (J.N.T.

$f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$0 < \theta < 1$

$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{\lfloor 3} \cdot f'''(\theta x)$

$\log(1+x) = 0 + 1 \cdot x + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3} \cdot \frac{2}{(1+\theta x)^3}$

$= x - \frac{x^2}{2} + \frac{x^3 \cdot 2}{3 \cdot 2 \cdot 1 (1+\theta x)^3}$

$x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$

$f''(x) = 2\tan x \sec^2 x = 2\tan x(1+\tan^2 x)$

(1) Maclaurin's series. *If $f(x)$ can be expanded as an infinite series, then*

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \quad (1)$$

If $f(x)$ possess derivatives of all orders and the remainder R_n in (3) on page 145 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example 4.20. *Using Maclaurin's series, expand $\tan x$ upto the term containing x^5.*

$f(x) = \tan x$, $f(0) = 0$

$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^5}{\lfloor 5} f^{v}(0)$

$f^{v}(0) = 16$

$\tan x = 0 + x \cdot 1 + \frac{x^3}{\lfloor 3} \cdot 2 + \frac{x^5}{\lfloor 5} 16$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5$

$(\sinh x)'$

$\sinh x = \cosh x$

$\cosh x = \sinh x$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$e^x \sin x$

Method 1

$f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \cdot \cos x$

$f(0) = 1$

$f'(x) = e^{\sin x} \cdot \cos x$

$f''(x) = e^{\sin x} \cdot \cos^2 x + e^{\sin x}(-\sin x)$ $\qquad f''(0) = 1$

$f'''(x) = \ldots$

(2) Expansion by use of known series. When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series :

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \ldots$
2. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \ldots$
3. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \ldots$
4. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \ldots$
5. $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2}{15}\theta^5 + \ldots$
6. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$
7. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$
8. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$
9. $\log(1-x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\right)$
10. $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$f^{iv}(0) = -3$

$f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \ldots$

$= 1 + x \cdot 1 + \dfrac{x^2}{2}$

Example 4.21. *Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.* (Bhopal, 2009; V.T

$= 1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8}$

Method 2

$e^{\sin x} = 1 + \sin x + \dfrac{(\sin x)^2}{2} + \dfrac{(\sin x)^3}{6} + \ldots$

$e^{\sin x} = 1 + \left(x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \ldots\right) + \dfrac{1}{2}\left(x - \dfrac{x^3}{6} - \ldots\right)^2 + \dfrac{1}{6}\left(x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \ldots\right)^3 + \dfrac{1}{24}\left(x - \dfrac{x^3}{6} - \ldots\right)^4$

$= 1 + x + \left(\dfrac{1}{2}\right)x^2 + \left(-\dfrac{1}{6} + \dfrac{1}{6}\right)x^3 + \left(-\dfrac{4}{24} \cdot \ldots + \dfrac{1}{24} - \dfrac{1}{6}\right)x^4$

$e^{\sin x} = 1 + x + \dfrac{x^2}{2} - \dfrac{1}{8}x^4$

$e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} - \ldots$

method-2

$\sin x = x - \dfrac{x^3}{6}$

$f(x) = e^{a\sin^{-1}x}$

f'

$f''(x) = e^{a\sin^{-1}x} \cdot a^2(1-x^2)^{-1} + e$

$= e^{a\sin^{-1}x} \cdot a^2(1-x^2)^{-1} + e^{a\sin^{-1}x}(1-x^2)^{-3/2} \cdot (ax)$

$f'''(x) = e^{a\sin^{-1}x} \cdot a^3(1-x^2)^{-3/2} + e^{a\sin^{-1}x} \cdot \left(\dfrac{-3}{2}\right)(1-x^2)^{-5/2}(-2x)(ax) + e^{a\sin^{-1}x}(1-x^2)^{-3/2} \cdot a$

$f'''(0) = a^3 + 0$

$e^{a\sin^{-1}x} = 1 + a\cdot x + \frac{x^2}{2}\cdot a^2 + \frac{x^3}{6} a(a^2+1)$

Example 4.23. *Expand* $e^{a\sin^{-1}x}$ *in ascending powers of x.*

$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) +$

(3) Taylor's series. *If* $f(x+h)$ *can be expanded as an infinite series, then*

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \infty \qquad \dots(1)$$

$x = a$

$h = (x-a)$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) on page 147, tends to zero as $n \to \infty$, then the Taylor's theorem becomes the *Taylor's series* (1).

Cor. Replacing x by a and h by $(x-a)$ in (1), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots \infty$$

Taking $a = 0$, we get *Maclaurin's series.*

$a = 1$ $\quad (x-2) \quad a = 2$

Example 4.24. *Expand* $\log_e x$ *in powers of* $(x-1)$ *and hence evaluate* $\log_e 1{\cdot}1$ *correct to 4 decimal places.*

$f(x) = \log x$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad f'''(1) = 2$

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a)$

$\log x = 0 + (x-1)\cdot 1 + (x-1)^2$

$\cdot(-1) + \frac{(x-1)^3}{6}\cdot 2 + \frac{(x-1)^4}{24}(-6)$

$\log x = (x-1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$ $\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$ 24 . .

$f^{IV}(x) = \frac{-6}{x^4}$ $f^{IV}(1) = -6$ $\log(1.1) = (0.1) + \frac{(0.1)^2}{2}(-1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$

Using Taylor's theorem, express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-1)$ (x^3)

Expand (i) e^x (*Cochin., 2005*) (ii) $\tan^{-1} x$, in powers of $(x-1)$ upto four terms.

$f(x) = 2x^3 + 7x^2 + x - 6 = a(x-1)^3 + b(x-1)^2 + c(x-1)^1 + d$

$f'(x) = 6x^2 + 14x + 1$

$f''(x) = 12x + 14$

$f'''(x) = 12$

$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2} f''(1)$ ———

$= 4 + (x-1)21 + \frac{x^2}{2} \cdot \frac{x-1}{6} \cdot 12$

$a = 2,\ b =\quad c = 21\quad d = 4$

Using Maclaurin's series, expand the following functions :

1. $\log(1+x)$. Hence deduce that $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

2. $\sin x$ (*P.T.U., 2005*)

3. $\sqrt{(1 + \sin 2x)}$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ ———

$\log\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right)$

$= \frac{1}{2}\left(\left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} - - -\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \frac{x^4}{4}\ —\right)\right)$

$\frac{1}{2}\left(2x + 2\frac{x^3}{3} + 2\frac{x^5}{5}\ —\right) = x + \frac{x^3}{3} + \frac{x^5}{5}$ ———

(3) $\sqrt{1 + \sin 2x}$ $f' = \frac{1}{2\sqrt{1+\sin 2x}} \cdot \cos 2x \cdot 2$

$f'' =$ (ч/v)

$\sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}$

$\sqrt{(\cos x + \sin x)^2} = \cos x + \sin x$

$f(x) = \cos x + \sin x$, $f'(x) = -\sin x + \cos x$ $f''(x) = -\cos x - \sin x$ f'

$\sqrt{1 + 2x} = 1 + x \cdot 1 + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3}(-1)$ ———

L-4 $\sin^{-1}x$

4. $\sin^{-1}x$ (Mumbai, 2007)
5. $\tan^{-1}x$
6. $\log \sec x$ (Mumbai, 2009 S ; V.T.U., 2009)

Prove that :

7. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots$
8. $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$ (Mumbai, 20
9. $\sin^{-1}\frac{2x}{1+x^2} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right\}$
10. $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x} = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$

$f(x) = \log \sec x \quad = 0 + x \cdot 0 + \frac{x^2}{\lfloor 2} 1 +$

$f'(x) = \frac{1}{\sec x} \cdot \sec x \cdot \tan x = \tan x$

$f''(x) = \sec^2 x = 1 + \tan^2 x$

$f'''(x) = 2 \cdot \tan x \cdot \sec^2 x$

$f^{iv}(x)$

$f^{v}(x) =$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) \qquad x = \tan\theta \qquad \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\theta = \tan^{-1}x$

$\sin^{-1}(\sin 2\theta)$

2θ

$f(x) = 2 \cdot \tan^{-1}x$

$= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \right)$

$\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right) \qquad$ Put $x = \tan\theta$

$\tan^{-1}\left(\frac{\sqrt{\sec^2\theta} - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \qquad 1 - \cos\theta$

$\tan^{-1}\left(\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$1 - \cos 2\theta =$

$\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$

$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \right)$

$$\tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots\right)$$

$0\cdot\infty$, $\infty-\infty$ $\quad 0^0$, 0^∞, 1^∞ $\quad \frac{0}{0}$, $\frac{\infty}{\infty}$

INDETERMINATE FORMS

$\underset{x\to 0}{Lt}\ \frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{0}{0}\times$

In general $\underset{x\to a}{\text{Lt}}\ [f(x)/\phi(x)] = \underset{x\to a}{\text{Lt}}\ f(x)/\underset{x\to a}{\text{Lt}}\ \phi(x)$. But when $\underset{x\to a}{\text{Lt}}\ f(x)$ and $\underset{x\to a}{\text{Lt}}\ \phi(x)$ are both zero, then the quotient reduces to the indeterminate form 0/0. This does not imply that $\underset{x\to a}{\text{Lt}}\ [f(x)/\phi(x)]$ is meaningless or it does not exist. In fact, in many cases, it has a finite value. We shall now, study the methods of evaluating the limits in such and similar other cases :

(1) $\underset{x\to 0}{Lt}\ \frac{\sin x}{x} = 1 \quad \frac{0}{0}$ $\quad \boxed{\frac{0}{0}\ \frac{\infty}{\infty}}$

Method-1 $\quad \underset{x\to 0}{Lt}\ \frac{f(x)}{g(x)} = \underset{x\to 0}{Lt}\ \frac{f'(x)}{g'(x)} = \underset{x\to 0}{Lt}\ \frac{f''(x)}{g''(x)} = \cdots\ \underset{x\to 0}{Lt}\ \frac{f^n(x)}{g^n(x)} = \text{Finite}$

$\underset{x\to 0}{Lt}\ \frac{\cos x}{1} = \frac{\cos(0)}{1} = \frac{1}{1} = 1$

$\underset{x\to 0}{Lt}\ \frac{\sin x}{\cos x} = \frac{0}{1} = 0$

Method-2 $\quad \underset{x\to 0}{Lt}\ \frac{\sin x}{x} = \underset{x\to 0}{Lt}\ \frac{\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)}{x} = \underset{x\to 0}{Lt}\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots\right) = 1$

$\frac{\infty}{\infty}$ $\frac{0}{0}$

$$\underset{x\to a}{\textbf{Lt}}\ \frac{\mathbf{f(x)}}{\mathbf{\phi(x)}} = \frac{\mathbf{f^n(a)}}{\mathbf{\phi^n(a)}} = \underset{x\to a}{\textbf{Lt}}\ \frac{\mathbf{f^n(x)}}{\mathbf{\phi^n(x)}} = \text{Fin}$$

[Rule to evaluate Lt [$f(x)/\phi(x)$] in 0/0 form :

Example 4.26. *Evaluate (i)* $\underset{x\to 0}{Lt}\ \dfrac{xe^x - \log(1+x)}{x^2}$.

(ii) $\underset{x\to 1}{Lt}\ \dfrac{x^x - x}{x-1-\log x}$

$x(1+\log x)$ $\quad x^x$

(1) $\frac{xe^x-\log(1+x)}{x^2} = \frac{x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}-\cdots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)}{x^2}$

M-2

$= \frac{\left(\cancel{x}+x^2+\frac{x^3}{2}+\frac{x^4}{6}-\cdots\right)-\left(\cancel{x}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)}{x^2}$

$\underset{x\to 0}{Lt}\ \frac{x^2\left(1+\frac{1}{2}\right)+x^3\left(\frac{1}{2}-\frac{1}{3}\right)-\cdots}{x^2} = \left(1+\frac{1}{2}\right)+x\left(\frac{1}{2}-\frac{1}{3}\right)$

$= \frac{3}{2}$

1. $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-b^x}{x}$ (*V.T.U., 2008*) **2.** $\underset{x\to 0}{\text{Lt}}\ \dfrac{x\cos x - \sin x}{x^2 \sin x}$

1. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$ (V.T.U., 2008)

2. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \sin x}$

3. $\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta - \sin \theta}{\sin \theta (1 - \cos \theta)}$

4. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log_e \sin x}$

(1) $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x} = \underset{x \to 0}{\text{Lt}} \dfrac{a^x \log a - b^x \log b}{1}$

$= \dfrac{\log a - \log b}{1} = \log\left(\dfrac{a}{b}\right)$

(1) $y = a^x$
$\log y = x \log a$
$\frac{1}{y} \cdot y' = \log a$
$y' = a^x \log a$

$\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta - \sin\theta}{\sin\theta(1-\cos\theta)} = \underset{\theta \to 0}{\text{Lt}} \dfrac{1 - \cos\theta}{\cos\theta(1-\cos\theta) + \sin\theta(\sin\theta)}$

$\dfrac{1-1}{0} = \dfrac{0}{0}$

$\dfrac{0}{0}$ $= \underset{\theta \to 0}{\text{Lt}} \dfrac{0 + \sin\theta}{-\sin\theta(1-\cos\theta) + \cos\theta(\sin\theta) + 2\sin\theta\cos\theta}$

$\sin^2$
$2\sin\theta\cos\theta$
$= \sin 2\theta$

$= \underset{\theta \to 0}{\text{Lt}} \dfrac{\cos\theta}{-\cos\theta(1-\cos\theta) - \sin\theta(\sin\theta) + \cos^2\theta - \sin^2\theta + \cos 2\theta \cdot 2}$

$= \dfrac{1}{1+2} = \dfrac{1}{3}$

$\underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log \sin x} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{a^{\sin x} \cdot \cos x \log a - 0}{\frac{1}{\sin x} \cdot \cos x}$

$y = a^{\sin x}$
$\log y = \sin x \cdot \log a$
$\frac{1}{y} \cdot y' = \cos x \cdot \log a$
$y' = a^{\sin x} \cdot \cos x \cdot \log a$

$= \dfrac{a^1 \cdot \log a}{1} = a \log a$

$\dfrac{0}{0}$ $\dfrac{\infty}{\infty}$ M-I

(2) Form ∞/∞. It can be shown that L'Hospital's rule can also be applied to this case by differentiating the numerator and denominator separately as many times as would be necessary.

$\dfrac{1}{\tan(0)}$

Example 4.28. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log x}{\cot x}$.

$\dfrac{\log(0)}{\cot(0)} = \dfrac{\infty}{\infty}$

$\underset{x \to 0}{\text{Lt}} \dfrac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = \underset{x \to 0}{\text{Lt}} \dfrac{-\sin^2 x}{x}$ $\dfrac{0}{0}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{-2\sin x \cos x}{1} = \dfrac{-2(0)1}{1} = 0$

Obs. Use of known series and standard limits. In many cases, it would be found more convenient to use expansions of known functions and standard limits for evaluating the indeterminate forms. For this purpose, remember the series of § 4.4 (2) and the following limits :

$$\boxed{\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1,} \qquad \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e \qquad (1^{\infty})$$

$$\underset{x \to 0}{Lt} (1+x)^{1/x} = e$$

$$\underset{x \to 0}{Lt} (1+x)^{1/x} = e$$

$$\underset{x \to 0}{Lt} (1+x^2)^{1/x^2} = e$$

$$\underset{x \to 0}{Lt} (1+3x^2)^{1/3x^2} = e$$

$$\underset{x \to 0}{Lt} \frac{\tan x}{x} = 1$$

$$y = (1+x)^{1/x}$$

$$\log y = \frac{1}{x} \log (1+x)$$

$$y = e^{\frac{1}{x} \log (1+x)}$$

$$y = e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} \text{———}\right)}$$

$$\underset{x \to 0}{Lt}\ y = e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} \text{———}\right)}$$

$$= e$$

M-1 M-2

Example 4.29. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log (1-x)}$.

$$\frac{0 - 0 - 0}{0 + 0} = \frac{0}{0}$$

H.W

Example 4.30. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{(1+x)^{1/x} - e}{x}$.

$$\underset{x \to 0}{Lt} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} \text{———}\right)} - e}{x}$$

$$e^{a+b} = e^a \cdot e^b$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \text{———}$$

$$= \frac{e^1 \cdot e^{\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right)} - e}{x}$$

$$\frac{e\left(1 + \left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right)^2 + \frac{1}{6}\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right)^3 \text{———}\right) - e}{x}$$

$$\frac{\cancel{e} + e\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right) + \frac{e}{2}\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right)^2 + \frac{e}{6}\left(-\frac{x}{2} + \frac{x^2}{3} \text{———}\right)^3 + \text{———}\right) - \cancel{e}}{x}$$

$$\underset{x \to 0}{Lt} \frac{\cancel{x}\left(e\left(-\frac{1}{2} + \frac{x}{3} - \frac{x^2}{}\right) + \frac{ex}{2}\left(-\frac{1}{2} + \frac{x}{3} \text{———}\right) + \frac{ex^2}{6}\left(-\frac{1}{2} + \frac{x}{3} \text{———}\right) \text{———}\right)}{\cancel{x}}$$

$$-\frac{e}{2}$$

$$\frac{0}{0} \quad \frac{\infty}{\infty}$$

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

I. Form 0 × ∞. If $\underset{x \to 0}{Lt} f(x) = 0$ and $\underset{x \to \infty}{Lt} \phi(x) = \infty$, then

$\underset{x \to a}{\text{Lt}} [f(x) . \phi(x)]$ assumes the form $0 \times \infty$.

To evaluate this limit, we write

$$f(x) \cdot \phi(x) = f(x)/[1/\phi(x)] \text{ to take the form } 0/0.$$
$$= \phi(x)/[1/f(x)] \text{ to take the form } \infty/\infty.$$

$\frac{0}{0}$ $\frac{\infty}{\infty}$

$\frac{1}{\infty}\infty \leftarrow 0.\infty \rightarrow 0.\frac{1}{0} = \frac{0}{0}$

$\frac{\infty}{\infty}$

$0.\infty$ $\frac{\infty}{\infty}$

Example 4.31. *Evaluate* $\underset{x \to 0}{Lt}$ *(tan x log x)*

$0.\infty$

$$\underset{x \to 0}{Lt} \frac{\log x}{\frac{1}{\tan x}} = \underset{x \to 0}{Lt} \frac{\log x}{\cot x} \quad \frac{\infty}{\infty}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = \underset{x \to 0}{Lt} \frac{-\sin^2 x}{x} \quad \frac{0}{0}$$

$$= \underset{x \to 0}{Lt} \frac{-2\sin x . \cos x}{1} = \frac{0}{1} = 0$$

$0.\infty$

$$\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1}{f(a)} - \frac{1}{g(a)} = \frac{g(a) - f(a)}{f(a) \cdot g(a)} = \frac{0-0}{0} = \frac{0}{0}$$

$\frac{0}{0}$ $\infty - \infty$

II. Form ∞ – ∞. If $\underset{x \to a}{Lt} f(x) = \infty = \underset{x \to a}{Lt} \phi(x)$, then $\underset{x \to a}{\text{Lt}} [f(x) - \phi(x)]$ assumes the form ∞ – ∞.

It can be reduced to the from 0/0 by writing

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\,\phi(x)}$$

Example 4.32. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

$\frac{1}{0} - \frac{1}{0}$

$\infty - \infty$

$\frac{\infty}{\infty}$ M-1

M-1

$\frac{0}{0}$ M-2

$$\underset{x \to 0}{Lt} \frac{x - \sin x}{x . \sin x} \quad \frac{0}{0}$$

$$\underset{x \to 0}{Lt} \frac{1 - \cos x}{x\cos x + \sin x} \quad \frac{0}{0}$$

$$\underset{x \to 0}{Lt} \frac{\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{0+1+1} = \frac{0}{2} = 0 \text{ Ans}$$

$0^0, 1^\infty, \infty^0$ $\underset{x \to a}{Lt} (f(x))^{\phi(x)}$

$0 \cdot \infty$

$$0^0 = e^{\log 0^0} = e^{0 \log 0} = e^{(0.\infty)}$$

$\left(0^0, 1^\infty, \infty^0\right)$ $\qquad \underset{x\to a}{Lt}\,(f(x))^{\phi(x)}$ $\qquad 0^0 = e^{\log 0^0} = e^{0 \log 0} = e^{(0\cdot\infty)}$

III. Forms 0^0, 1^∞, ∞^0. If $y = \underset{x\to a}{\text{Lt}}\,[f(x)]^{\phi(x)}$ assumes one of these forms, then $\log y = \underset{x\to a}{\text{Lt}}\,\phi(x)\,\log f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in I above. If $\log y = l$, then $y = e^l$.

Example 4.33. *Evaluate* (*i*) $\underset{x\to\pi/2}{Lt}\,(\sin x)^{\tan x}$ (*ii*) $\underset{x\to 0}{Lt}\left(\dfrac{a^x + b^x + c^x}{3}\right)^{1/x}$

$\frac{0}{0}, \frac{\infty}{\infty}$ H.W (*iii*) $\underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$ $\quad 1^\infty$

$e^{\log f(x)} = f(x)$

$\log\left(e^{f(x)}\right) = f(x)$

(1) $\underset{x\to\frac{\pi}{2}}{Lt}(\sin x)^{\tan x} = \underset{x\to\frac{\pi}{2}}{Lt}\, e^{\log(\sin x)^{\tan x}} = \boxed{e^0 = 1 \text{ Ans}}$

$(\infty \cdot 0)$

$\underset{x\to\frac{\pi}{2}}{Lt}\log(\sin x)^{\tan x} = \underset{x\to\frac{\pi}{2}}{Lt}\tan x \cdot \log \sin x$

$= \underset{x\to\frac{\pi}{2}}{Lt}\dfrac{\log \sin x}{\cot x}$ $\qquad \frac{0}{0}$

$= \dfrac{\frac{1}{\sin x}\cdot\cos x}{-\operatorname{cosec}^2 x} = \underset{x\to\frac{\pi}{2}}{Lt}\dfrac{-\sin^2 x}{\sin x}\cdot\cos x = -1\cdot 0 = 0$

1^∞

$\underset{x\to 0}{Lt}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x}$

(a) $\log(abc)^{1/3}$
(b) $\frac{1}{3}(abc)$
(c) $(abc)^3$
(d) $(abc)^{1/3}$

e^b

$\underset{x\to 0}{Lt}\, e^{\log\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}} = e^{\underset{x\to 0}{Lt}\frac{1}{x}\left(\log(a^x+b^x+c^x) - \log 3\right)}$

$\left(\frac{0}{0}\right)$ $\underset{x\to 0}{Lt}\dfrac{\log(a^x+b^x+c^x) - \log 3}{x} = \dfrac{1}{a^x+b^x+c^x}\left(a^x\log a + b^x\log b + c^x\log c\right) - 0$

$= \dfrac{1}{1+1+1}(\log a + \log b + \log c) = \frac{1}{3}\log(abc) = \log(abc)^{1/3}$

$= e^{\log(abc)^{1/3}} = (abc)^{1/3}$

$\boxed{\underset{x\to a}{Lt}\, a^b = e^{\underset{x\to a}{Lt}(a-1)\cdot b}}$

$e^{\left(\frac{a^x+b^x+c^x}{3} - 1\right)\cdot\frac{1}{x}}$

$= e^{\frac{1}{3}\left(a^x\log a + b^x\log b + c^x\log c\right)}$

III $\underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$

$\underset{x\to 0}{Lt}\left(\dfrac{x + \frac{x^3}{3} + \frac{2}{15}x^5 - }{x}\right)^{1/x^2}$

$\underset{x\to 0}{Lt}\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + -\right)^{1/x^2}$

$Lt\left(1 + x^2\left(\frac{1}{3} + \frac{2}{15}x^2 + -\right)\right)^{1/x^2}$

$\boxed{\begin{aligned} &\underset{x\to 0}{Lt}(1+x)^{1/x} = e \\ &\underset{x\to 0}{Lt}(1+x^2)^{1/x^2} = e \\ &\underset{x\to 0}{Lt}(1+t\cdot x^2)^{1/tx^2} = e \end{aligned}}$

$$\lim_{x\to 0}\left(1+x^2\left(\frac{1}{3}+\frac{2}{15}x^2+\ldots\right)\right)^{1/x^2}$$

$$\lim_{x\to 0}\left(1+x^2t\right)^{1/x^2}$$

$$\lim_{x\to 0}\left(\left(1+x^2t\right)^{1/x^2t}\right)^{t} = e^{\lim t} = e^{1/3}$$

Ans

$$t = \frac{1}{3}+\frac{2}{15}x^2+\ldots$$

$$\frac{1}{3}$$

$\infty-\infty$ $\qquad\qquad$ $\frac{1}{0}-\frac{1}{0}$

1. $\displaystyle \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ $\qquad$ **2.** $\displaystyle \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

3. $\displaystyle \lim_{x\to \pi/2}(2x\tan x-\pi\sec x)$ (*V.T.U., 2008*) $\qquad$ **4.** $\displaystyle \lim_{x\to 0}\left(\frac{\cot x-1/x}{x}\right)$

$\frac{0}{0}$ L-Hosp

$$\lim_{x\to\frac{\pi}{2}}\frac{2x\cdot\sin x}{\cos x}-\pi\cdot\frac{1}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{2x\sin x-\pi}{\cos x} = \frac{2\sin x+2x\cos x-0}{-\sin x}$$

$$= \frac{2+0}{-1} = -2$$

(4)
$$\lim_{x\to 0}\left(\frac{\cot x-1/x}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{x\sin x}-\frac{1}{x^2}\right)$$

$$= \lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}$$

$$= \lim_{x\to 0}\frac{-x\sin x+\cancel{\cos x}-\cancel{\cos x}}{x^2\cos x+2x\sin x}$$

$$= \lim_{x\to 0}\frac{-\sin x}{x\cos x+2\sin x} = \frac{-\cos x}{-x\sin x+\cos x+2\cos x} = \frac{-1}{3}$$

5. $\displaystyle \lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ $\quad$ $\infty-\infty$ $\qquad$ **6.** $\displaystyle \lim_{x\to 1}(x)^{1/(1-x)}$

7. $\displaystyle \lim_{x\to 0}(a^x+x)^{1/x}$ (*V.T.U., 2007*) $\qquad$ **8.** $\displaystyle \lim_{x\to \pi/2}(\sec x)^{\cot x}$

9. $\displaystyle \lim_{x\to 0}(1+\sin x)^{\cot x}$ $\qquad$ **10.** $\displaystyle \lim_{x\to 0}(\cos x)^{1/x^2}$

11. $\displaystyle \lim_{x\to \pi/2}(\tan x)^{\tan 2x}$ (*V.T.U., 2004*) $\qquad$ **12.** $\displaystyle \lim_{x\to 0}(\cot x)^{1/\log x}$

a b c $\qquad$ $0-0$

Example 4.27. *Find the values of a and b such that* $\displaystyle Lt\ \frac{x(a+b\cos x)-c\sin x}{\ldots}=1.$

Example 4.27. *Find the values of a and b such that* $\underset{x\to 0}{Lt}\ \dfrac{x(a+b\cos x)-c\sin x}{x^5}=1.$

$$\frac{0}{0}$$

$$\underset{x\to 0}{Lt}\ \frac{x(-b\sin x)+(a+b\cos x)-c\cos x}{5x^4}=1$$

$$\frac{0}{0} \qquad \frac{a+b-c}{0}=1 \qquad \boxed{a+b-c=0} \text{ --- (1)}$$

$$\frac{0}{0} \qquad \underset{x\to 0}{Lt}\ \frac{(-b\sin x)+x(-b\cos x)+(-b\sin x)+c\sin x}{20x^3}=1$$

$$a+60-180=0$$

$$\boxed{a=120}$$

$$\underset{x\to 0}{Lt}\ \frac{(-2b+c)\sin x-bx\cos x}{20x^3}$$

$$\underset{x\to 0}{Lt}\ \frac{(-2b+c)\cos x+bx\sin x-b\cos x}{60x^2}=1$$

$$\frac{-2b+c-b}{0}=1$$

$$\boxed{-3b+c=0}$$

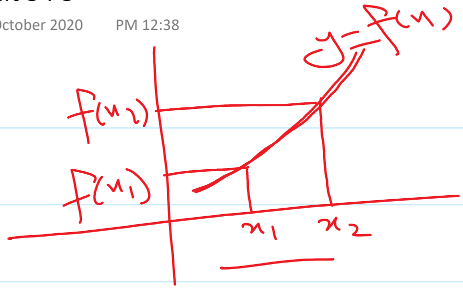
$$c=3b$$

$$\boxed{c=180}$$

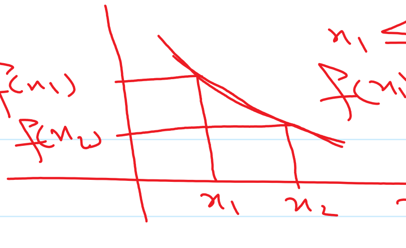
$$\underset{x\to 0}{Lt}\ \frac{(-2b+\cancel{3b})^{b}\cos x+bx\sin x-\cancel{b\cos x}}{60x^2}=1$$

$$\underset{x\to 0}{Lt}\ \frac{b\cancel{x}\sin x}{60x^{\cancel{2}}}=\frac{b}{60}\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)=1$$

$$=\frac{b}{60}=1 \Rightarrow \boxed{b=60}$$



$x_1 \le x_2$ <
$f(x_1) \le f(x_2)$ <
Inc
str Inc



$x_1 \le x_2$ <
$f(x_1) \ge f(x_2)$ >
st Dec
Dec

Increasing and decreasing function

1.2.4 Increasing and Decreasing Functions

Let $y = f(x)$ be a function defined on an interval I contained in the domain of the function $f(x)$. Let x_1, x_2 be any two points in I, where x_1, x_2 are not the end points of the interval. On the interval I, the function $f(x)$ is said to be

(i) an *increasing* function, if $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$.

(ii) a *strictly* increasing function, if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(iii) a *decreasing* function, if $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$.

(iv) a *strictly* decreasing function, if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

A function which is either increasing or decreasing in the entire interval I is called a *monotonic* function.

$f(x) \pm 2$

f'

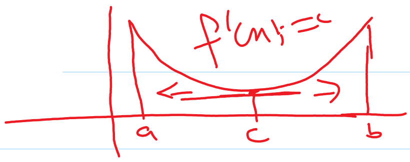
$\frac{e^{\sin x} - \cos x}{e^x}$

$f(x)$
$f'(x) > 0 \quad 2x < 0 \quad \forall x \in I$

Therefore, we conclude that

(i) f increases in I if $f'(x) > 0$ for all x in I.

(ii) f decreases in I if $f'(x) < 0$ for all x in I.



Example 1.10 Find the intervals in which the function $f(x) = \sin 3x$, $0 \le x \le \pi/2$ is increasing or decreasing.

$f(x) = \sin 3x$

$f'(x) = 3\cos 3x = 0$

$\cos 3x = 0$

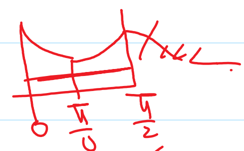
$3x = (2n+1)\frac{\pi}{2}$

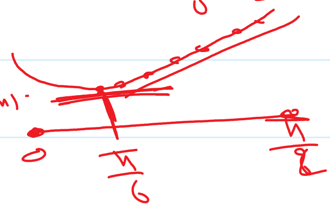
$x = (2n+1)\frac{\pi}{6}$

$x = \frac{\pi}{6}$

$\cos\theta = 0$
$\theta = (2n+1)\frac{\pi}{2}$
$n = 0, \pm 1, \pm 2$
$\cos\theta = 0$
$\theta = (2n+1)\frac{\pi}{2}$

$0 \le x \le \frac{\pi}{2}$





$0 \le x \le \frac{\pi}{2}$

+ve

in $0 \le x \le \frac{\pi}{6}$ $\quad f'(x) = 3\cos 3x = 3 \cdot 1 = 3$ +ve Inc.

Dec

$\frac{\pi}{6} \le x \le \pi$ $\quad f'(x) = 3\cdot\cos 3\frac{\pi}{2} = 3\cos(\pi - \frac{\pi}{2}) = -3\cos\frac{\pi}{2}$ -ve

$0 \le x \le \frac{\pi}{4}$ $\quad f'(x) = 3\cos 3x = 3.(+ve) = +ve$ Inc

dec $\frac{\pi}{6} \le x \le \frac{\pi}{2}$ $\quad f'(x) = 3.\cos\frac{3\pi}{4} = 3\cos(\pi - \frac{\pi}{4}) = -3\cos\frac{\pi}{4}$ $-ve$ Dec

$\frac{\pi}{4}$

29. $\ln(2+x) - 2x/(2+x), \quad x \in \mathbb{R}$

30. $x|x|, \quad x \in \mathbb{R}$.

$f(x) = \log(2+x) - \frac{2x}{(2+x)} \qquad x \in \mathbb{R} \quad (-\infty, \infty)$

$f'(x) = \frac{1}{2+x} - \frac{(2+x)(2) - 2x(1)}{(2+x)^2} = \frac{1}{2+x} - \frac{4 + \cancel{2x} - \cancel{2x}}{(2+x)^2}$

$= \frac{1}{2+x} - \frac{4}{(2+x)^2} = \frac{2+x-4}{(2+x)^2} = \frac{x-2}{(2+x)^2}$

$x = 2$

$(-\infty, \infty)$

$(-\infty, 2) \quad (2, \infty)$

$-ve \qquad +ve$

Dec $\qquad$ Inc

30 $\quad x|x| \quad x \neq 0 \quad x \neq 0$ $\quad \frac{d}{dx}(|x|) = \frac{x}{|x|}$

$f'(x) = x.\frac{x}{|x|} + |x| = \frac{x^2 + |x|^2}{|x|} = +ve$ Inc $(-\infty, \infty)$

31. $\tan^{-1} x + x, \quad x \in \mathbb{R}$.

$\frac{1}{1+x^2} + 1$ $\qquad$ (~~a~~) Inc

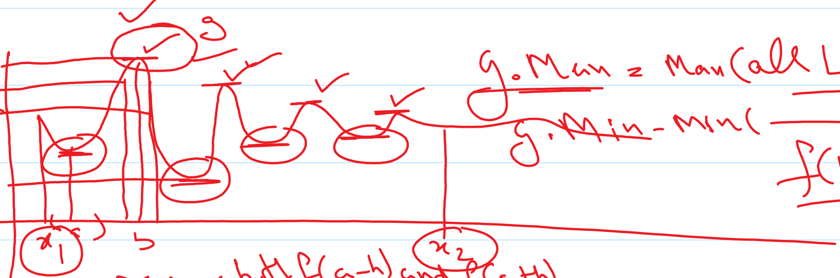
(b) Dec

$\frac{1 + 1 + x^2}{1+x^2} = \frac{2+x^2}{1+x^2} = +ve$ Inc $(-\infty, \infty)$ $\forall x \in \mathbb{R}$

$\tan^{-1}x - x$

$\frac{1}{1+x^2} - 1 = \frac{1 - 1 - x^2}{1+x^2} = \frac{-x^2}{1+x^2}$ $+ve$ $\quad -ve$ Dec

$(-\infty, \infty)$



$f'(x) = 0$

$x = a, b, < f(b+h)$

g.Max = Max(all L.Max)

g.~~Min~~-Min(

$f(b) >$ both $f(b-h)$ and $f(b+h)$

$x = a$

$f(a) <$ both $f(a-h)$ and $f(a+h)$

$x = a$

$(a-h, a+h)$ x_1 x_2

$f(a) <$ both $f(a-h)$ and $f(a+h)$

MAXIMA AND MINIMA

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h) and f(a + h).*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

(3) Procedure for finding maxima and minima

$f(x)$ (*i*) *Put the given function = f(x)*

(*ii*) *Find f′(x) and equate it to zero. Solve this equation and let its roots be a, b, c, ...*

$f'(x) = 0$ (*iii*) *Find f″(x) and substitute in it by turns x = a, b, c, ...*

$x = a, b, c, \ldots$ *If f″(a) is – ve, f(x) is maximum at x = a.*

If f″(a) is + ve, f″(x) is minima at x = a.

$f''(x) =$

(*iv*) Sometimes *f″*(*x*) may be difficult to find out or *f″*(*x*) may be zero at *x* = *a*. In such cases, see if *f′*(*x*) *changes sign from + ve to – ve as x passes through a, then f(x) is maximum at x = a.*

If f′(x) changes sign from – ve to + ve as x passes through a, f(x) is minimum at x = a.

If f′(x) does not change sign while passing through x = a, f(x) is neither maximum nor minimum at x = a.

f' $a-h$ a $a+h$ +ve −ve

$f''(x)$ $f''(a) = 0$

$\ldots f^{(n)}(a) \neq 0$

3

$f^{(4)}(a) \neq 0$

$f^{IV}(a) \neq 0$ +ve −ve

Theorem 1.6 Let $f^{(n)}(x)$ exist for x in (a, b) and be continuous there. Let

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0.$$

$f'''(x) \neq 0$

Then, $n \geq 3$

(i) when n is even, $f(x)$ has a maximum if $f^{(n)}(x_0) < 0$ and a minimum if $f^{(n)}(x_0) < 0$

(ii) when n is odd, $f(x)$ has neither a maximum, nor a minimum.

global

Absolute maximum/minimum values of a function $f(x)$ in an interval $[a, b]$ are defined as follows:

Absolute maximum value = max {$f(a)$, $f(b)$, all local maximum values}.

Absolute minimum value = min {$f(a)$, $f(b)$, all local minimum values}.

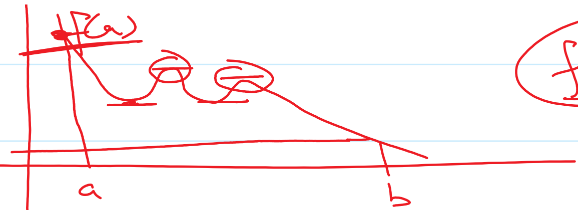
$f(a)$

$f'(x) = 0$

$f(1) = 2$

$f(-1) = -6$

$f(0) = -8$



$f'(x) = 0$

$[a, b]$

$f(a)$ ✓

$f(b)$ ✓

$2, -6, 1$

-6

f

$[0, 2]$

Example 4.56. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

$f(0) = 1$

$f(2) =$

$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$f'(x) = 12x^3 - 6x^2 - 12x + 6 = 0$

$2x^3 - x^2 - 2x + 1 = 0$

$x^2(2x-1) - 1(2x-1) = 0$

$(x^2 - 1)(2x-1) = 0$

$f''(x) = 36x^2 - 12x - 12$

$f''(1) = 36 - 12 - 12 = +ve$ min

$f''(-1) = 36 + 12 - 12 = +ve$ min

$f''(\frac{1}{2}) = 9 - 6 - 12 = -ve$ max

$x = \pm 1, \frac{1}{2}$

Min(

$f(1) = 3 - 2 - 6 + 6 + 1 = 2$ ✓

$f(-1) = 3 + 2 - 6 - 6 + 1 = -6$ ✓

$f(\frac{1}{2}) = \frac{3}{16} - \frac{2}{8} - \frac{6}{4} + \frac{6}{2} + 1$

$\frac{3 - 4 - 24 + 48 + 16}{16} = \frac{39}{16}$

Example 4.57. *Show that sin x (1 + cos x) is a maximum when x = π/3.*

Example 1.13 Find the absolute maximum/minimum values of the function

$f(x) = \sin x(1 + \cos x), \quad 0 \le x \le 2\pi.$

$[0, 2\pi]$

$f'(x) = \cos x(1 + \cos x) + \sin x(-\sin x)$

$= \cos x + \cos^2 x - \sin^2 x = 0$

$= \cos x + \cos 2x = 0$

$= \cos x + 2\cos^2 x - 1 = 0$

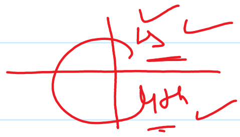
$= 2\cos^2 x + \cos x - 1 = 0$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2x = 2\cos^2 x - 1$

$\cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$

$f'(x) = \cos x + \cos 2x$



$\cos x = -1$

$x = \pi$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$2\pi - \frac{\pi}{3}$

$2\pi + \frac{\pi}{3}$

x zero.

$\cos \pi = (-1)^n$

$\cos 0 = 1$

$\sin 0 = 0$

$x = \pi$

$f''(x) = -\sin x - 2\sin 2x$

$f''(\pi) = -\sin(\pi) - 2\sin(2\pi) = 0$

$f'(x) = -\sin x - 2\sin 2x$

$f''(\pi) = -\sin(\pi) - 2\sin(2\pi) = 0$

$f'''(\pi) = -\cos\pi - 4\cos 2\pi = 1 - 4 = -3 \neq 0$

at $x = \pi$ no min no max

$2\frac{\pi}{3} = \pi - \frac{\pi}{3}$

$x = \frac{\pi}{3}$ $\quad f''(x) = -\sin x - 2\sin 2x$

$f''(x) = -\sin\frac{\pi}{3} - 2\sin 2\frac{\pi}{3} = -\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} = -ve$ Max

$x = \frac{5\pi}{3}$ $\quad f''\left(\frac{5\pi}{3}\right) = -\sin\frac{5\pi}{3} - 2\sin\left(\frac{10\pi}{3}\right) = \frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2} = +ve$ min $\quad 3\pi + \frac{\pi}{3}$

$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)$ ✓

$f(x) = \sin x(1 + \cos x)$

$f\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right)$ ✓

$g.\text{Max} = \left\{-\frac{\sqrt{3}}{2}\left(\frac{3}{2}\right), 0\right\}$

$f(0) = 0$ ✓

$f(2\pi) = 0$ ✓

$f'(x) = 0 \quad x = ?$

38. $(x-1)^2(x+1)^3$. **39.** $\sin x + \cos x$.

40. $x^{1/x}$. **41.** $(\sin x)^{\sin x}$

$f(x) = (x-1)^2(x+1)^3$

$f'(x) = 2(x-1)(x+1)^3 + (x-1)^2 3(x+1)^2$

$= (x-1)(x+1)^2(2(x+1) + 3(x-1)) = 0$

$(x-1)(x+1)^2(5x-1) = 0$

$x = 1, \; x = -1, \; x = \frac{1}{5}$

$\sin x + \cos x$

$f'(x) = \cos x - \sin x = 0$

$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4} + n\pi$

$n = 0, \pm 1, \pm 2$

$x^{\frac{1}{x}} = e^{\log x^{\frac{1}{x}}}$

$\left(e^{f(x)}\right)' = e^{f(x)} \cdot (f(x))'$

$f(x) = e^{\frac{1}{x}\log x}$

$f'(x) = e^{\frac{1}{x}\log x}\left(\frac{1}{x}\cdot\frac{1}{x} + \left(-\frac{1}{x^2}\right)\cdot\log x\right) = \underbrace{e^{\frac{1}{x}\log x}}_{\neq 0}\left(\frac{1-\log x}{x^2}\right) = 0$

$\frac{1-\log x}{x^2} = 0 \Rightarrow 1 - \log x = 0 \quad \log x = 1$

$\boxed{x = e}$

$(\sin x)^{\sin x} = e^{\log(\sin x)^{\sin x}} = e^{\sin x \log \sin x}$

$\underbrace{e^{\sin x \log(\sin x)}}_{\neq 0}\left(\cancel{\sin x}\cdot\frac{1}{\cancel{\sin x}}\cdot\cos x + \cos x \log \sin x\right)$

$\cos x(1 + \log \sin x) = 0$

$\cos x = 0 \quad \log \sin x = -1$

$\cos x(1+\log \sin x) = 0$

$\cos x = 0$ | $\log \sin x = -1$

$\boxed{x = (2n+1)\frac{\pi}{2}}$ | $\sin x = e^{-1}$

$n = 0, \pm 1, \pm 2$ — | $x = \sin^{-1}(e^{-1})$

Example 1.14 Find a right angled triangle of maximum area with hypotenuse h.

Solution Let x be the base of the right angled triangle. The area of the right angled triangle is

$f'(x) = 0$

$$A(x) = \frac{1}{2}\, x\, \sqrt{h^2 - x^2}\,, \quad 0 < x < h.$$

$\frac{1}{2}\left(\sqrt{h^2-x^2} + x \cdot \frac{1}{2\sqrt{h^2-x^2}} \cdot (-2x)\right)$

Now,

$$A'(x) = \frac{1}{2}\left[\sqrt{h^2 - x^2} - \frac{x^2}{\sqrt{h^2 - x^2}}\right] = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}} = 0$$

$h^2 - 2x^2 = 0$

$x = \pm \frac{h}{\sqrt{2}}$

$x = \frac{h}{\sqrt{2}}$

Setting $A'(x) = 0$, we obtain the critical point as $x = h/\sqrt{2}$.

$h^2 - 2\frac{h^2}{2}$ $\quad x_0 = \frac{h}{\sqrt{2}}$ $\quad f'(x_0)$

$h^2 - \cancel{2}\frac{h^2}{\cancel{2}}$

$\frac{h^2 - 2x^2}{2\sqrt{h^2-x^2}}$ +ve

$x = \frac{h}{\sqrt{2}}$

$a-h$ $\quad a$ $\quad a+h$

+ve $\quad$ −ve $\quad$ Max

−ve $\quad$ +ve $\quad$ Min

$x = \frac{h}{\sqrt{2}}$ Max

+ve $\quad x < \frac{h}{\sqrt{2}}$ $\quad a-h$

$x > \frac{h}{\sqrt{2}}$ −ve

$f' \to$ +ve

$2 - x$ $\quad 0$

$x = 2 \quad x < 2$

Now, $A'(x) > 0$ for $x < h/\sqrt{2}$ and $A'(x) < 0$ for $x > h/\sqrt{2}$.

Therefore, $A(x)$ is maximum when $x = h/\sqrt{2}$ and the maximum area is $A(h/\sqrt{2}) = h^2/4$.